One interesting inequality.

https://www.linkedin.com/feed/update/urn:li:activity:6535402651998392320

Let $x_1, x_2, \ldots, x_n$ be any real numbers. Prove that

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \ldots + \frac{x_n}{1+x_1^2+x_2^2+\ldots+x_n^2} < \sqrt{n}.$$

Solution by Arkady Alt, **San Jose**, **California**, **USA**.

Since $\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \ldots + \frac{x_n}{1+x_1^2+x_2^2+\ldots+x_n^2} \le$

$$\frac{|x_1|}{1+x_1^2} + \frac{|x_2|}{1+x_1^2+x_2^2} + \ldots + \frac{|x_n|}{1+x_1^2+x_2^2+\ldots+x_n^2}$$

suffice to prove inequality of the problem for $x_1, x_2, \ldots, x_n \ge 0$.

Let $a_k := \frac{x_k}{1+x_1^2+x_2^2+\ldots+x_k^2}, k = 1,2,\ldots,n$ and $S_n := \sum_{k=1}^{n} a_k$.

By Cauchy Inequality $S_n \le \sqrt{n \cdot \sum_{k=1}^{n} a_k^2}$.

Noting that $1+x_1^2+x_2^2+\ldots+x_{k-1}^2 < 1+x_1^2+x_2^2+\ldots+x_{k-1}^2+x_k^2$ we obtain

$$a_k^2 = \frac{x_k^2}{(1+x_1^2+x_2^2+\ldots+x_k^2)^2} < \frac{x_k^2}{(1+x_1^2+x_2^2+\ldots+x_{k-1}^2)(1+x_1^2+x_2^2+\ldots+x_k^2)} =$$

$$\frac{1}{1+x_1^2+x_2^2+\ldots+x_{k-1}^2} - \frac{1}{1+x_1^2+x_2^2+\ldots+x_k^2}$$ and, therefore,

$$\sum_{k=1}^{n} a_k^2 < \sum_{k=1}^{n}\left(\frac{1}{1+x_1^2+x_2^2+\ldots+x_{k-1}^2} - \frac{1}{1+x_1^2+x_2^2+\ldots+x_k^2}\right) = 1 - \frac{1}{1+x_1^2+x_2^2+\ldots+x_n^2} < 1.$$

Thus, $S_n < \sqrt{n}$.